Simply Transitive Primitive Groups Whose Maximal Subgroup Contains a Transitive Constituent of Order p^2, or pq, or a Transitive Constituent of Degree 5.

By Elizabeth Ruth Bennett.

If the maximal subgroup G_1 of degree $n-1$ of a simply transitive primitive group G of degree n contains a transitive constituent of degree 3, it is known that it must also contain another transitive constituent of degree 3 or 6.* Similar restrictions on G_1 have been determined when G_1 contains a transitive constituent of degree 4.† We proceed to consider restrictions that may be placed on the degree of G or on the transitive constituents of G_1 when G_1 contains a transitive constituent of order p^2 or pq, or a transitive constituent of degree 5.

Theorem I. *If a transitive constituent of G_1 is of order p^2, the degree of G is q^a, q a prime. When k, the number of transitive constituents of order p^2 in G_1, is odd, $k \equiv 3$, mod. 4, and $q=2$. When $k \equiv 2$, mod. 4, q is a prime of form $4n+3$ and a is an odd number.*

All groups of order p^2 are abelian and can be represented transitively only in regular form. Therefore the order of G_1 can not exceed p^2 and G_1 must be formed from the simple isomorphism of groups of degree and order p^2.‡ G is then of class $n-1$ and of degree q^a, q a prime.§

The following relation must exist where k represents the number of transitive constituents of order p^2 in G_1:

$$kp^2+1=q^a. \tag{I}$$

By considering (I) with respect to modulus 4, the remainder of the theorem is evident.

Theorem II. *If a transitive constituent of G_1 is of order pq, p and q primes, $p > q$, and the order of G_1 is pq, G_1 is formed from establishing a*

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* Miller, American Journal of Mathematics, Vol. XXXV, p. 7.

† Bennett, American Journal of Mathematics, Vol. XXXIV, pp. 8, 9.

‡ Miller, *Bulletin American Mathematical Society,* Vol. VI, p. 104.

§ Frobenius, *Berliner Sitzungsberichte* (1902), pp. 455–459.

simple isomorphism between groups of order pq. When the constituent of order pq is abelian, G is of degree r^a, r a prime, and of class $n-1$. If the order of G_1 exceeds pq, G_1 must contain a transitive constituent of degree pq whose order is greater than pq, and, in case the constituent of order pq is of degree p, the order of G_1 must be $q^a p$.

When the order of G_1 is pq, G_1 is formed from the simple isomorphism of groups of order pq, for the order of G_1 is not divisible by the square of a prime number.*

A group of order pq can be represented transitively only on pq or p letters, and, in case the group of order pq is abelian, the representation must be on pq letters. Therefore, if the order of G_1 is pq and the constituent of order pq is abelian, G is of class $n-1$ and degree r^a, r a prime. When the order of G_1 exceeds pq and the transitive constituent of order pq is regular, then G_1 must contain an additional transitive constituent of degree pq whose order exceeds pq.† When the constituent of order pq is of degree p, the subgroup leaving a letter of the constituent of degree p fixed is composed of transitive constituents of order and degree q. G_1 must then contain a transitive constituent whose degree is pq and whose order is greater than pq.‡ If the constituent of order pq is of degree p, the order of G_1 must be $q^a p$,† for the order of G_1 is not divisible by p^2.

COROLLARY I. *If G_1 contains a dihedral group of prime degree p as a transitive constituent, the order of G_1 is $2^a p$.*

THEOREM III. *When G_1 contains k transitive constituents of order 5, k an odd number, the degree of G is $2^{4\beta}$ and $k \equiv 3$, mod. 4. If $k \equiv 2$, mod. 4, the degree of G is q^a, q a prime of form $4n+3$ and a an odd number.*

Since G_1 contains a transitive constituent of order 5, the order of G_1 is 5 and G is of class $n-1$. The degree of G is then q^a, q a prime. If the number of transitive constituents of order 5 is odd, $5k+1$ is even or $5k+1=2^a$ and $k \equiv 3$, mod. 4. Since 2 is a primitive root of 5, in order that the above equation be satisfied $a=4\beta$. If $k \equiv 2$, mod. 4, $5k+1 \equiv 3$, mod. 4, and in order that $q^a \equiv 3$, mod. 4, a must be odd and q a prime of form $4n+\beta$.

THEOREM IV. *If G_1 contains the semi-metacyclic group of degree 5 as a transitive constituent, G_1 must contain another transitive constituent of degree 10 or 5 and the order of G_1 must be $2^a \cdot 5$.*

* Miller, *Proceedings London Mathematical Society*, Vol. XXVIII, p. 536.

† Reitz, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI, p. 9.

‡ Bennett, *loc. cit.*, p. 6.

When the order of G_1 is 10, G_1 can be formed only from the simple isomorphism of groups of degrees 5 and 10. If the order of G_1 exceeds 10, from Theorem II G_1 must contain a transitive constituent of degree 10, and the order of G_1 is $2^\alpha \cdot 5$.

THEOREM V. *If G_1 contains the alternating group of degree 5 as a transitive constituent and the order of G_1 is 60, G_1 is formed from the simple isomorphism of groups whose degree can be only 60, 30, 20, 15, 12, 10, 6 and 5. If the order of G_1 exceeds 60, G_1 must contain a transitive constituent of degree 20. The order of G_1 is $2^\alpha \cdot 3^\beta \cdot 5$.*

When the order of G_1 is 60, since the alternating group of degree 5 in simple, the order of the other transitive constituents of G_1 must also be 60. The group of order 60 can be represented only on 60, 30, 20, 15, 12, 10, 6 and 5 letters; therefore, only transitive constituents of such degrees may occur when G_1 is of order 60. If the order of G_1 exceeds 60, G_1 must contain a transitive constituent of degree 20, for the subgroup leaving fixed a letter of the alternating group of degree 5 is a primitive group.* Since the order of G_1 is not divisible by 5^2, the order of G is $2^\alpha \cdot 3^\beta \cdot 5$.

A theorem concerning the symmetric group of degree 5 may be stated which differs from Theorem V only with regard to the possible representations.

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* Bennett, *loc. cit.*, p. 6.